선형대수학 (Linear Algebra)

1 Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix}$. Find the matrices S and K, where $A = S + K$, S is symmetric, and K is skew-symmetric.

2 Show that if A is an $n \times n$ nonsingular matrix, then the inverse of A is unique.

3 Show that if A is a diagonal matrix with nonzero diagonal entries $a_{11},\ a_{22}, \cdots, a_{nn}$, then A is nonsingular and A^{-1} is a diagonal matrix with diagonal entries $\frac{1}{a_{11}},\ \frac{1}{a_{22}}, \cdots, \frac{1}{a_{nn}}$.

4 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. Find A^{-1}.

5 Show that if A and B are both nonsingular $n \times n$ matrices, then AB is also nonsingular.

6 Let A and B be $n \times n$ matrices. Show that if AB is nonsingular, then A and B must be nonsingular.

7 Show that if A is an $n \times n$ nonsingular matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

8 Show that if A is an $n \times n$ matrix and the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$, then A is nonsingular.

9 Show that if $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{u} - \mathbf{v}$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

10 Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a set of vectors in a vector space V, then span S is a subspace of V.

11 Let S_1 and S_2 be finite subsets of a vector space and $S_1 \subset S_2$. Show that if S_1 is linearly dependent, so is S_2.

12 Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2.

13 Let $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ belong to the solution space of $A\mathbf{x} = \mathbf{0}$. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent? Explain.

14 Find a basis for the subspace W of R^3 spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}.$$

What is the dimension of W?

15 Show that if two finite-dimensional vector spaces are isomorphic, then their dimensions are equal.

16 Show that if A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n.$$

17 Show that if A is an $n \times n$ matrix and the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$, then rank $A = n$.

18 Show that if A is an $n \times n$ matrix and rank $A = n$, then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

19 Show that if A is an $n \times n$ matrix and the columns of A are linearly independent, then rank $A = n$.

20 Show that if A is an $n \times n$ matrix and rank $A = n$, then the columns of A are linearly independent.

21 Let $S = \{\mathbf{v}_1,\ \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1,\ \mathbf{w}_2\}$ be ordered bases for R^2, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If the transition matrix from S to T is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, determine T.

22 Compute the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}.$$

23 Let $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} a \\ -1 \\ -b \end{bmatrix}$. For what values of a and b is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal set?

24 Use the Gram-Schmidt process to transform the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

for the Euclidean space R^3 into an orthonormal basis for R^3.

25 Let $S = \{\mathbf{u}_1,\ \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an orthonormal basis for a Euclidean space V and let $\mathbf{v}$ be any vector in V. Show that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n,\ \text{where } c_i = (\mathbf{v}_i, \mathbf{u}_i),\ i = 1, 2, \cdots, n.$$

26 Let W be a subspace of an inner product space V, then show that $W^\perp$ is a subspace of V and $W \cap W^\perp = \{\mathbf{0}\}$.

27 Show that if W is a finite-dimensional subspace of an inner product space V, then $(W^\perp)^\perp = W$.

28 Let W be the subspace of R^4 spanned by

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ x-1 \\ 4 \end{bmatrix} \right\}.$$

Find a basis for $W^\perp$.

29 Show that if $L : V \to W$ is a linear transformation of a vector space V into a vector space W, then ker L is a subspace of V.

30 Show that if $L : V \to W$ is a linear transformation of an n-dimensional vector space V into a vector space W, then

$$\dim \ker\ L + \dim \operatorname{range}\ L = \dim V.$$

31 Show that if A and B are similar $n \times n$ matrices, then rank $A =$ rank B.

32 Show that if A and B are similar $n \times n$ matrices and A is nonsingular, then B is also nonsingular.

33 Show that if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

34 Show that if A and B are similar $n \times n$ matrices, then $\det A = \det B$.

35 Is $\det(AB) = \det(BA)$? Justify your answer.

36 Compute the area of the parallelogram with vertices $(2,3),\ (5,3),\ (4,5),\ (7,5)$.

37 Find the inverse matrix of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

after compute $\det A$ and $\mathrm{adj} A$.

38 Find the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

39 Show that similar matrices have the same eigenvalues.

40 Find a 3×3 nondiagonal matrix whose eigenvalues are $-2,\ -2,\ 3$ and associated eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, respectively.