Abstract Algebra Problems

1. Prove that if $\phi : X \to Y$ is an isomorphism of $\langle X, * \rangle$ with $\langle Y, \cdot \rangle$, then ϕ^{-1} is isomorphism of $\langle Y, \cdot \rangle$ with $\langle X, * \rangle$.

2. Let n be a positive integer and let $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$. Show that $\langle n\mathbb{Z}, + \rangle \simeq \langle \mathbb{Z}, + \rangle$.

3. Show that every group G with identity e and such that $a^2 = e$ for all $a \in G$ is abelian.

4. Give a complete list of all subgroup relations, of the form $G_i \leq G_j$, that exist between these given groups G_1, G_2, G_3, where

$G_1 = \mathbb{Z}$ under addition

$G_2 = 12\mathbb{Z}$ under addition

$G_3 = \mathbb{R}$ under addition

5. Find the order of the cyclic subgroup of the given group generated by the indicated element.

(1) The subgroup of $\mathbb{Z}_{12}$ generated by 9

(2) The subgroup of $\mathbb{Z}_4$ generated by 3

6. Show that if H and K are subgroups of an abelian group G, then $\{hk \mid h \in H \text{ and } k \in K\}$ is a subgroup of G.

7. Show that a nonempty subset H of the group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

8. Prove or disprove that Every finite group of at most four elements is abelian.

9. Prove that every cyclic group is abelian.

10. Show that a group with no proper nontrivial subgroups is cyclic.

11. Show that the followings are not true by using a counterexample or by giving a short argument.

(a) If a group G is such that every proper subgroup is cyclic, then G is cyclic.

(b) All generators of $\mathbb{Z}_{20}$ are prime numbers.

(c) S_n is not cyclic for any n.

12. Let a and b be elements of a group G. Show that if ab has finite order n, then ba has order n.

13. Let G be an abelian group and let H and K be finite cyclic subgroups with $|H| = r$ and $|K| = s$.

(a) Show that if r and s are relatively prime, then G contains a cyclic subgroup of order rs.

(b) Generalizing part (a), show that G contains a cyclic subgroup of order which is the least common multiple of r and s.

14. Show that $\mathbb{Z}_p$ has no proper nontrivial subgroups if p is a prime number.

15. Let p and q be prime numbers. Find the number of generators of the cyclic group $\mathbb{Z}_{pq}$.

16. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

17. Compute the indicated product involving the following permutations in S_6.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}.$$

(a) $|\langle \sigma \rangle|$

(b) $|\langle \tau^2 \rangle|$

(c) σ^{100}

(d) μ^{100}

18. Consider S_n for a fixed $n \geq 2$ and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.

19. Show that if σ is a cycle of odd length, then σ^2 is a cycle of odd length.

20. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

21. Find all cosets of the subgroup $\langle 18 \rangle$ of $\mathbb{Z}_{36}$.

22. Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

23. Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$.

24. Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.

25. Let H be a subgroup of a group G and let $g \in G$. Define a one-to-one map of H onto Hg. Prove that your map is one to one and is onto Hg.

26. Show that if H is a subgroup of index 2 in a finite group G, then every left coset of H is also a right coset of H.

27. Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_6$.

28. Prove that a direct product of abelian groups is abelian.

29. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

30. Prove that if a finite abelian group has order a power of a prime p, then the order of every element in the group is a power of p. Can the hypothesis of commutativity be dropped? Why, or why not?

31. Determine whether the given map ϕ is a homomorphism.
Let $\phi : \mathbb{Z}_6 \to \mathbb{Z}_2$ be given by $\phi(x) =$ the remainder of x when divided by 2, as in the division algorithm.

32. Compute $\ker(\phi)$ and $\phi(3, 10)$ for a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to S_{10}$ where $\phi(1, 0) = (3\ \ 5)(2\ \ 4)$ and $\phi(0, 1) = (1\ \ 7)(6\ \ 10\ \ 8\ \ 9)$.

33. Give an example of a nontrivial homomorphism $\phi : D_4 \to S_3$, if an example exists. If no such homomorphism exists, explain why that is so.

34. Let G be a group. Let $h, k \in G$ and let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be defined by $\phi(m, n) = h^m k^n$. Give a necessary and sufficient condition, involving h and k, for ϕ to be a homomorphism. Prove your condition.

35. Show directly from the definition of a normal subgroup that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H.

36. Let K and L be normal subgroups of G with $K \vee L = G$ and $K \cap L = \{e\}$. show that $G/K \cong L$ and $G/L \cong K$.

37. Let G and H be groups and let $\phi : G \to H$ be a homomorphism. Prove that the image of ϕ, $\phi(G)$, is a subgroup of H. Prove that if ϕ is injective then $G \cong \phi(G)$.

38. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some n} \in \mathbb{Z}^+\}$. Prove that for any fixed integer $k > 1$, the map from G to itself defined by $\mathbb{Z} \mapsto \mathbb{Z}^k$ is a surjective homomorphism but is not an isomorphism.

39. Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \leq (A \times B)$ is normal and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

40. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

41. Prove that if G is a nonabelian group with $|G| = 8$, then group G is **not** simple.

42. Let $\phi : (G, *) \to (\mathbb{Z}, +)$ be a homomorphism and N containing $\ker \phi$ be a subgroup of G. Then N is normal subgroup of G.

43. Prove that if a finite group G has exactly one subgroup H of order d, then H is normal subgroup of G.

44. Let H and N be subgroup of G. Show by an example that $G/H \not\simeq G/K$ and $H \simeq K$.

45. Let H is subgroup of group G. Define $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove that H is a normal subgroup of G if and only if $N_G(H) = G$.

46. Let $\phi : (\mathbb{Z}_{30}, +) \to (\mathbb{Z}_6, +)$ be a homomorphism with $\ker \phi = \{0, 6, 12, 18, 24\}$. Show that ϕ is surjective and find all possible value of $\phi(1)$.

47. Let $\phi : G \to G'$ be a homomorphism and N' be a normal subgroup of G'. Show that $\phi^{-1}(N')$ is a normal subgroup of G.

48. If H and K are normal subgroup of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H, k \in K$.

49. Let N be a normal subgroup of G and let H be a subgroup of G. Let $HN = \{hn \mid h \in H, n \in N\}$. Show that HN is subgroup of G, and is the smallest subgroup containing H and N.

50. Let H be subgroup of G and N be the normal subgroup of G. Show that by an example that $H \cap N$ need not be normal in G.